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Spectroscopy of cosmic topology

Tarun Souradeep

Inter-University Centre for Astronomy and Astrophysics, Post Bag 4, Ganeshkhind, Pune-411 007, Maharastra, India

E-mail tarun@iucaa ernet in

Abstract : Einstein's theory of gravitation that governs the geometry of space-time, coupled with spectacular advance in cosmological observations, promises to deliver a 'standard model' of cosmology in the near future However, local geometry of space constrains, but does not dictate the topology of the cosmos Hence, cosmic topology has remained an enigmatic aspect of the 'standard model' of cosmology Recent advance in the quantity and quality of observations has brought this issue within the realm of observational query The breakdown of statistical homogeneity and isotropy of cosmic perturbations is a generic consequence of non trivial cosmic topology arising from the imposed 'crystallographic' periodicity on the eigenstates of the Laplacian The sky maps of Cosmic Microwave Background (CMB) anisotropy and polarization are the most promising observations that could carry signatures of a violation of statistical isotropy and homogeneity Hence, a *measurable* spectroscopy of cosmic topology is made possible using the Bipolar power spectrum (BiPS) of the temperature and polarization that quantifies violation of statistical isotropy

Keywords : Cosmic microwave background-cosmology theory, observations, topology

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...ted honored to be invited to contribute to this volume ...noring the memory of *Professor Amal Kumar ...ichaudhuri* (fondly known as AKR) – a great scientist ...d teacher The *Raychaudhuri equations* describing the ...olution of anisotropic universe models are the footprints ...homegrown Indian science in the field of cosmology ...hough the background universe is observationally ...nsistent with homogeneous and isotropic Friedmann ...odels, the Raychaudhuri equations appears in the ...volution of inhomogeneities that led to the formation of ...rge scale structures in the universe. It is fair to say much ...f the recent progress in cosmology has come from the ...terplay between refinement of the theories of structure ...rmation and the improvement of the observations. Hence, ...e Raychaudhuri equations have remained as relevant ...nd ingrained in contemporary cosmology as when first ...ut forward by AKR. This article that describes our ongoing ...search to determine the topology of the universe from ...xquisite measurements of anisotropy in the Cosmic Microwave background is a humble tribute to the doyen ...f Indian science.

The realization that a universe with the same local geometry has many different choices of global topology has been a theoretical curiosity as old as modern cosmology De Sitter was quick to point out that the first modern model of the cosmos, Einstein's closed (S^3, spherical geometry and static) universe model, could equally well correspond to the multiply connected 'Elliptical' universe where antipodal point of S^3 are topologically identified (S^3/Z_2). Figure 1 depicts the prevalent modern view within the concept of inflation, that this relatively smooth 'Hubble volume' that we observe is perhaps a tiny patch of an extremely inhomogeneous and complex spatial manifold The complexity could involve non-trivial topology (multiple connectivity) on these ultra-large scales Given the observational support for a homogeneous Hubble volume around us, the diverse possibility of global structure reduces to the tractable study limited to spaces of uniform curvature (locally homogeneous and isotropic FLRW models) but with non-trivial cosmic topology. For example, in Figure 1, observers in the 'handle' regions would perceive an open (hyperbolic geometry) universe, and those in the 'bulb' region observe a closed (spherical geometry) universe. Although, in a generic manifold,

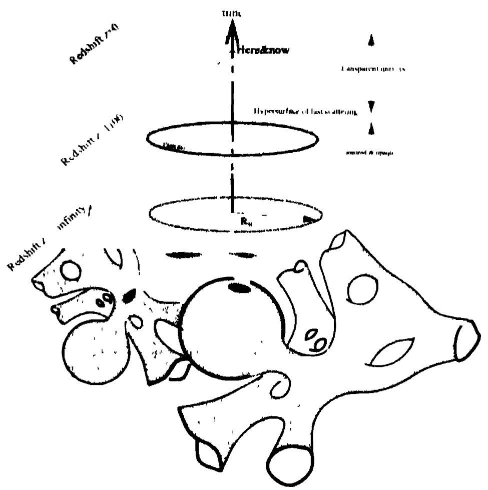

Figure 1 A cartoon depicting a prevalent view within the inflationary paradigm The observable universe corresponds to a small patch of a very complicated manifold that has been blown to cosmological scales during an inflationary epoch Ultra-large scale structure could be observable if the size of this patch is not much smaller that the scales of inhomogeneity and non-trivial topology

sectors with exact Euclidean geometry are rare, an epoch of inflation in the early universe inflate any region to a nearly flat, Euclidean geometry

The motivation for non-trivial topology and the quest to determine size and the shape of our universe has a rich and diverse history in modern cosmology [1–5] Spatially compact universe models have both deep theoretical and philosophical appeal – *e.g.*, the quantum creation of finite sized universe from vacuum is a theoretical motivation [6], and a philosophical abhorrence of any 'infinity' in nature would argue against an infinite sized universe [7] A compact universe, with the exception of the three sphere, S^3 implies a multiply connected universe (*i e.*, nontrivial cosmic topology)

The photons of the cosmic microwave background (CMB) propagate freely over distances comparable to the cosmic horizon The CMB anisotropy and its polarization are the most promising observational probes of the global spatial structure of the universe on length scales near to and even somewhat beyond the 'horizon' scale ($\sim cH_0^{-1}$). ***A generic consequence*** [36] ***of cosmic topology is the breaking of statistical isotropy in characteristic patterns determined by the photon geodesic structure of the manifold*** [9] The increasingly exquisite measurements of CMB anisotropy have brought cosmic topology from the realm of theoretical possibility to within the grasp of observations and has received considerable attention over the past few years [9–15] CMB polarization not only augments the CMB temperature anisotropy observations but is expected to allow a more incisive study of cosmic topology since it arises only at the surface of last scattering (SLS) The recent full sky measurements of CMB polarization maps by WMAP has opened the door to this new arena [17]

Any multiply-connected space is equivalent to a simply connected space (universal cover) that is tiled by Dirichlet domains (DD) under a free acting subgroup, Γ of the isotropy group, G^u of universal cover ***Hence, even the most complicated cosmic topology reduces to a study of standard FLRW models with periodic boundary condition on the appropriate DD.*** A simple example being the 3-torus T^3, that is equivalent to the study of a universe in a box with periodic boundary conditions – a routine approximation in numerical simulations of large scale structure in the universe More formally the T^3 is obtained by tiling 3-D Euclidean space under the isometry subgroup of discrete translations in three directions In cosmology the Dirichlet domain constructed around the observer represents the universe as 'seen' by the observer The SI breakdown is apparent in the principal axes present in the shape of the DD constructed with the observer located at the base-point [18] Equivalently, the fields defining perturbations are built of the subset of eigenstates of Laplacian that invariant under Γ The correlations of CMB fluctuations in the sky would have patterns that are no longer invariant under rotations

In a cosmological model with trivial topology the CMB anisotropy signal is expected to be statistically isotropic *i.e.*, statistical expectation values of the temperature fluctuations $\Delta T(\hat{q})$ are preserved under rotations of the sky. In particular, the angular correlation function $C(\hat{q},\hat{q}') \equiv \langle \Delta T(\hat{q}) \Delta T(\hat{q}') \rangle$ is rotationally invariant for Gaussian fields, *i.e.*, $C(\hat{q},\hat{q}') \equiv C(\hat{q}\cdot\hat{q}')$. In spherical harmonic space, where $\Delta T(\hat{q}) = \sum_{lm} a_{lm} Y_{lm}(\hat{q})$, condition of *statistical isotropy* (SI) translates to a diagonal $\langle a_{lm} a^*_{l'm'} \rangle = C_l \delta_{ll'} \delta_{mm'}$ where C_l, the widely used power spectrum of CMB anisotropy The correlation in the CMB anisotropy that lead to violation of SI implies that imply $\langle \hat{a}_{lm} \hat{a}^*_{lm} \rangle$ has off-diagonal elements Figure taken from [13] shows the off-diagonal elements in the CMB correlation for two compact universe models

The observed CMB sky $\widetilde{\Delta T}(\hat{n})$ is a single realization

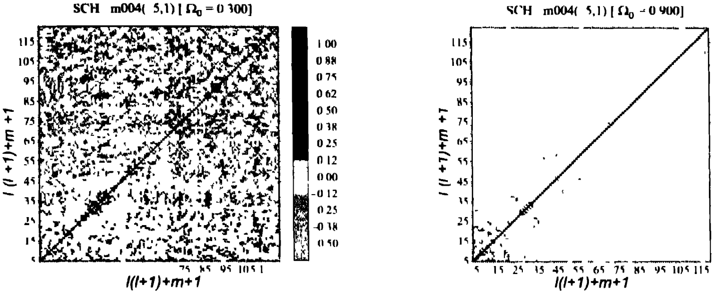

Figure 2 The Figure taken from [13] illustrates the non-diagonal nature of the expectation values of $a_{\ell m}$ pair products when the CMB anisotropy violates SI in two model compact universe The radical violation in the model on the left corresponds to a small compact universe where CMB photons have traversed across multiple times The model on the left with mild violation of SI corresponds to a universe of size comparable to the observable horizon For more details, see [13].

the underlying correlation, hence the detection of SI violation or correlation patterns pose a great observational challenge [37] For statistically isotropic CMB sky, the correlation function

$$C(\hat{n}_1,\hat{n}_2) \equiv C(\hat{n}_1 \cdot \hat{n}_2) = \frac{1}{8\pi^2}\int d\mathcal{R}\, C(\mathcal{R}\hat{n}_1,\mathcal{R}\hat{n}_2) \qquad (1)$$

where $\mathcal{R}\hat{n}$ denotes the direction obtained under the action of a rotation $\mathcal{R}$ on $\hat{n}$, and $d\mathcal{R}$ is a volume element of the three-dimensional rotation group The invariance of the underlying statistics under rotation allows the estimation of $C(\hat{n}_1 \cdot \hat{n}_2)$ using the average of the temperature product $\Delta T(\hat{n})\Delta T(\hat{n}')$ between all pairs of pixels with the angular separation θ. In the absence of statistical isotropy, $C(\hat{n}_1,\hat{n}_2)$ is estimated by a single product $\widetilde{\Delta T}(\hat{n})\widetilde{\Delta T}(\hat{n}')$ and hence, is poorly determined from a single realization

Although it is not possible to estimate each element of the full correlation function $C(\hat{n}_1,\hat{n}_2)$, some measures of statistical anisotropy of the CMB map can be estimated through suitably weighted angular averages of $\widetilde{\Delta T}(\hat{n})\widetilde{\Delta T}(\hat{n}')$. The angular averaging procedure should be such that the measure involves averaging over sufficient number of independent 'measurements', but should ensure that the averaging does not erase all the signature of statistical anisotropy Recently, we proposed the Bipolar Power spectrum (BiPS) κ_ℓ ($\ell = 1, 2, 3,\ldots$) of the CMB map as a statistical tool of detecting and measuring departure from SI [18] The BiPS is formally defined as

$$\kappa^\ell = (2l+1)^2 \int d\Omega_{n_1} \int d\Omega_{n_2}$$

$$\times\left[\frac{1}{8\pi^2}\int d\mathcal{R}\,\chi^\ell(\mathcal{R})C(\mathcal{R}\hat{n}_1,\mathcal{R}\hat{n}_2)\right]^2 \qquad (2)$$

In the above expression, $C(\mathcal{R}\hat{n}_1,\mathcal{R}\hat{n}_2)$ is the two point correlation at $\mathcal{R}\hat{n}_1$ and $\mathcal{R}\hat{n}_2$ which are the coordinates of the two pixels $\hat{n}_1$ and $\hat{n}_2$ after rotating the coordinate system by element $\mathcal{R}$ of the rotation group

$\chi^\ell(\mathcal{R})$ is the trace of the finite rotation matrix in the ℓM-representation

$$\chi^\ell(\mathcal{R}) = \sum^{\ell} D^\ell_{MM}(\mathcal{R}), \qquad (3)$$

which is called the *characteristic function*, or the character of the irreducible representation of rank ℓ It is invariant under rotations of the coordinate systems in eq (2) $d\mathcal{R}$ is the volume element of the three-dimensional rotation group For a statistically isotropic model $C(\hat{n}_1,\hat{n}_2)$ is invariant under rotation, and therefore $C(\mathcal{R}\hat{n}_1,\mathcal{R}\hat{n}_2) = C(\hat{n}_1,\hat{n}_2)$ and the orthonormality of $\chi^\ell(\omega)$, we will recover the condition for SI,

$$\kappa^\ell = \kappa^0\delta_{\ell 0} \qquad (4)$$

The Bipolar power spectrum gets it name from its interpretation in the harmonic space The two point correlation of CMB anisotropies, $C(\hat{n}_1,\hat{n}_2)$, is a two point function on $S^2 \times S^2$, and hence can be expanded as

$$C(\hat{n}_1,\hat{n}_2) = \sum_{l_1,l_2,L,\ldots} A^{\ell M}_{l_1 l_2}\left\{Y_{l_1}(\hat{n}_1)\otimes Y_{l_2}(\hat{n}_2)\right\}_{\ell M}, \qquad (5)$$

where $A^{\ell M}_{l_1 l_2}$ are coefficients of the expansion (here after BipoSH coefficients) and $\left\{Y_{l_1}(\hat{n}_1)\otimes Y_{l_2}(\hat{n}_2)\right\}_{\ell M}$ are the Bipolar spherical harmonics which transform as a spherical harmonic with ℓ, M with respect to rotations [20] given by

$$\left\{Y_{l_1}(\hat{n}_1)\otimes Y_{l_2}(\hat{n}_2)\right\}_{\ell M} = \sum C^{\ell M}_{l_1 m_1 l_2 m_2} Y_{l_1 m_1}(\hat{n}_1) Y_{l_2 m_2}(\hat{n}_2), \qquad (6)$$

in which $C^{\ell M}_{l_1 m_1 l_2 m_2}$ are Clebsch-Gordan coefficients We can inverse-transform $C(\hat{n}_1, \hat{n}_2)$ to get the $A^{\ell M}_{l_1 l_2}$ by multiplying both sides of eq (5) by $\{Y_{l_1'}(\hat{n}_1) \otimes Y_{l_2'}(\hat{n}_2)\}_{\ell' M'}$ and integrating over all angles, then the orthonormality of bipolar harmonics implies that

$$A^{\ell M}_{l_1 l_2} = \int d\Omega_{\hat{n}_1} \int d\Omega_{\hat{n}_2} C(\hat{n}_1, \hat{n}_2) \{Y_{l_1}(\hat{n}_1) \otimes Y_{l_2}(\hat{n}_2)\}^*_{\ell M} \quad (7)$$

The above expression and the fact that $C(\hat{n}_1, \hat{n}_2)$ is symmetric under the exchange of $\hat{n}_1$ and $\hat{n}_2$ lead to the following symmetries of $A^{\ell M}_{l_1 l_2}$

$$A^{\ell M}_{l_2 l_1} = (-1)^{(l_1 + l_2 + l)} A^{\ell M}_{l_1 l_2},$$

$$A^{\ell M}_{ll} = A^{\ell M}_{ll} \delta_{\ell, 2k}, \; k = 0, 1, 2, \ldots \quad (8)$$

The Bipolar Spherical Harmonic (BipoSH) coefficients, $A^{\ell M}_{l_1 l_2}$, are linear combinations of off-diagonal elements of the harmonic space covariance matrix,

$$A^{\ell M}_{l_1 l_2} = \sum \left\langle a_{l_1 m_1} a^*_{l_2 m_2} \right\rangle (-1)^{m_2} C^{\ell M}_{l_1 m_1 l_2 - m_2} \quad (9)$$

This means that $A^{\ell M}_{l_1 l_2}$ completely represent the information of the covariance matrix in harmonic space $\left\langle a_{l_1 m_1} a^*_{l_2 m_2} \right\rangle$ When SI holds, the harmonic space covariance matrix is diagonal and hence

$$A^{\ell M}_{ll'} = (-1) C_l (2l+1)^{1/2} \delta_{ll'} \delta_{\ell 0} \delta_{M 0},$$

$$A^{00}_{l_1 l_2} = (-1)^{l_1} \sqrt{2 l_1 + 1}\, C_{l_1} \delta_{l_1 l_2}. \quad (10)$$

BipoSH expansion is the most general representation of the two point correlation functions of CMB anisotropy. The well known angular power spectrum, C_l is a subspace of BipoSH coefficients corresponding to the A^{00}_{ll} that represent the statistically isotropic part of a general correlation function When SI holds, A^{00}_{ll} or equivalently C_l have all the information of the field But when SI breaks down, A^{00}_{ll} are not adequate for describing the field, and one needs to take the other terms into account The Bipolar power spectrum (BiPS) is defined as a rotationally invariant contraction of the BipoSH coefficients

$$\kappa_\ell = \sum_{l, l', M} \left| A^{\ell M}_{ll'} \right|^2 \geq 0 \quad (11)$$

This definition is identical to the real space expression in eq (2). More importantly, BiPS is measurable from a single CMB map since averages over many independent modes and reduces the cosmic variance [38] [19]. The BiPS of the CMB anisotropy maps measured by WMAP has been computed [21] Preliminary BiPS results on the CMB polarization maps from the three year of WMAP data have also emerged in past few months [22,23]

The BiPS is sensitive to structures and patterns in the underlying total two-point correlation function [19,24] The BiPS is particularly sensitive to real space correlation patterns (preferred directions, *etc*) on characteristic angular scales In harmonic space, the BiPS at multipole ℓ sums power in off-diagonal elements of the covariance matrix, $\langle a_{lm} a_{l'm'} \rangle$, in the same way that the 'angular momentum' addition of states $lm, l'm'$ have non-zero overlap with a state with angular momentum $|l - l'| < \ell < l + l'$. Signatures like a_{lm} and $a_{l+n m}$ being correlated over a significant range l are ideal targets for BiPS These are typical of SI violation due to cosmic topology and the predicted BiPS in these models have a strong spectral signature in the bipolar multipole ℓ space [18] The orientation independence of BiPS is an advantage since one can obtain constraints on cosmic topology that do not depend on the unknown (but specific) orientation of the pattern (*e g.*, preferred directions of DD relative to the sky)

Spaces of constant curvature have been completely classified [25,26]. For Euclidean geometry, there are known to be six possible topologies that lead to orientable spaces The simple flat torus, $\mathcal{M} = T^3$, is obtained by identifying the universal cover $\mathcal{M}^u = \mathcal{E}^3$ under a discrete group of translations along three non-degenerate axes, s_1, s_2, s_3 $\rightarrow s_i + nL_i$, where L_i is the identification length of the torus along s_i and n is a vector with integer components In the most general form, the fundamental domain (FD) is a parallelepiped defined by three sides L_i and the three angles α_i between the axes ('squeezed torus') If is an orthogonal then one gets cuboid FD, which for equal l reduces to the cubic torus The cuboid and squeezed spaces which can be obtained by a linear coordinate transformation $\mathcal{L}$ on cubic torus can have distinctly different global symmetry [39].

Study of the BiPS signature of cosmic topology has already been undertaken and ongoing [18,27,28] The correlation function $C(\hat{q}, \hat{q}')$ for CMB anisotropy in multiply-connected universe such as the torus space can be computed using the regularized method of images [12, Figure 3 plots the predicted κ_ℓ spectrum for a number of cubic, cuboid and squeezed torus spaces [18]. Similar results for Poincaré Dodecahedron show a characteristic BiPS with a dominant peak at κ_6 over a range of value of curvature radius (including integrated Sachs-Wolfe effect) This relates to the angular separation of the directions to faces of the extremely symmetric DD of Poincaré dodecahedral space Figure 3 shows that κ_ℓ is zero for odd ℓ

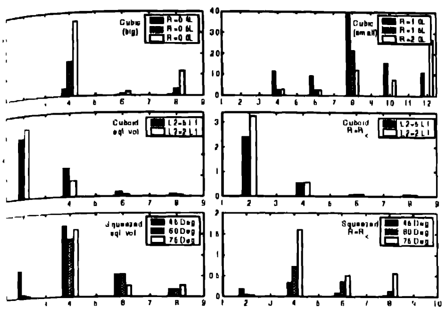

Figure 3 The Figure taken from [18] shows the BiPS κ_ℓ spectra for flat torus models The top row panels are for cubic tori spaces The left panel shows spaces of volume, V_M, larger than the volume V_*, contained in the sphere of last scattering (SLS) with $V_M/V_* = 3.7, 1.9, 1.1$, respectively The right panel shows small spaces with $V_M/V_* = 0.24$, 0.03 respectively Note that $\kappa_2 = 0$ for cubic tori The middle panels consider cuboid tori with 1 5 and 1 2 ratio of identification lengths The bottom panels show κ_ℓ for equal-sided squeezed tori with α 45°, 60° and 75° In the middle and bottom rows, the right panels show the case when radius of SLS, $R_* = R_<$ the in-radius of the space i.e. the SLS just touches its nearest images which is at the threshold where CMB anisotropy is multiply imaged for larger R_*. The cases in the left panels of lower two rows have $V_M/V_* = 1$ and are at the divide between large and small spaces

This is intimately related to the symmetries of the Dirichlet domain which in turn is dictated by the properties of the subgroup of isometries Γ The BiPS is zero at odd bipolar multipoles if DD has $2n$-fold symmetry about an axis and reflection symmetry in an orthogonal plane It can be proved that all Euclidean and all spherical spaces generated by a single action Γ satisfy this condition Remarkably enough, compact hyperbolic spaces do not satisfy these condition, and are generically expected to have non-zero BiPS at odd value bipolar multipoles ℓ [29] This provides a *measurable* classification of cosmic topology based on CMB anisotropy and polarization, *i e.*, ***a spectroscopy of cosmic topology.***

A simple working example is the BiPS signature of a non-trivial topology can be given for a T^3 universe, where the correlation function is given by

$$C(\hat{q},\hat{q}') = L^3 \sum P_\Phi(k_n) e^{i\pi(\epsilon_{\hat{q}} n \cdot \hat{q} - \epsilon_{\hat{q}'} n \cdot \hat{q}')}, \tag{12}$$

in which, $\boldsymbol{n}$ is 3-tuple of integers (in order to avoid confusion, we use $\hat{q}$ to represent the direction instead of $\hat{n}$), the small parameter $\epsilon_{\hat{q}} \leq 1$ is the physical distance to the SLS along $\hat{q}$ in units of $L/2$ (more generally, $\bar{L}/2$ where $\bar{L} = (L_1 L_2 L_3)^{1/3}$) and L is the size of the Dirichlet domain (DD) When ϵ is a small constant, the leading order terms in the correlation function eq (12) can be readily obtained in power series expansion in powers of ϵ For the lowest wave numbers $|\boldsymbol{n}|^2 = 1$ in a cuboid torus [18]

$$C(\hat{q},\hat{q}') = 2\sum_i P_\Phi(2\pi/L_i)\cos(\pi\epsilon\, \beta_i \Delta q_i)$$

$$\approx C_0\left[1 - \epsilon^2 |\Delta q|^2 + 3\epsilon^4 \sum_{i=1} (\Delta q_i)^4\right], \tag{13}$$

where Δq_i are the components of $\Delta q = \hat{q} - \hat{q}'$ along the three axes of the torus and $\beta_i = \bar{L}/L_i$ From this, the non-zero κ_ℓ can be analytically computed to be

$$\frac{\kappa_0}{C_0^2} = \pi^2\left(1 - 4\epsilon^2 + \frac{368}{15}\epsilon^4 - \frac{288}{5}\epsilon^6 + \frac{20736}{125}\epsilon^8\right)$$

$$\frac{\kappa_4}{C_0^2} = \frac{12288\pi^2}{875}\epsilon^8 \tag{14}$$

κ_4 has the information of the relative size of the Dirichlet domain and one can use it to constrain the topology of the universe

The results of WMAP are a milestone in CMB anisotropy measurements since it combines high angular resolution, high sensitivity, with 'full' sky coverage allowed by a space mission The *Wilkinson Microwave Anisotropy Probe (WMAP)* observations are consistent with the predictions of the concordance ΛCDM model with scale-invariant and adiabatic fluctuations which have been generated during the inflationary epoch [30–32] After the first year of WMAP data, the SI of the CMB anisotropy (*i.e* rotational invariance of n-point correlations) has attracted considerable attention Tantalizing evidence of SI breakdown (albeit, in very different guises) that mounted in the WMAP first year sky maps, using a variety of different statistics are expected to persist in the three persist in three year data (see [23] for discussion and references)

The CMB anisotropy map based on the WMAP data are ideal for testing for statistical isotropy Preferred directions and statistically anisotropic CMB anisotropy have been discussed in literature earlier [33,34] A number of direct searches for signature of cosmic topology have been proposed and carried out on early CMB data from COBE-DMR Full Bayesian likelihood comparison to the data of specific cosmic topology models is another approach that has applied to COBE-DMR data [9,12,13] The generic features of κ_ℓ spectrum are related to the symmetries of correlation pattern For cosmic topology, κ_ℓ are sensitive to SI violation even when CMB is not multiply imaged The orientation independence of BiPS is an advantage for constraining patterns (preferred directions) with unspecified orientation in the CMB sky such as that arising due to cosmic topology or, anisotropic cosmology [35] Extension of BiPS analysis to CMB polarization maps has been studied recently [22,23] adds a new dimension to the spectroscopy of cosmic topology

In summary, there are strong theoretical and philosophical motivations for a non-trivial cosmic topology The breakdown of statistical homogeneity and isotropy of cosmic perturbations is a generic feature of non trivial cosmic topology A promising observational approach is to hunt for SI violation in the CMB anisotropy The underlying correlation patterns in the CMB anisotropy and polarization in a multiply connected universe is related to the symmetry of the Dirichlet domain BiPS has the advantage of being independent of the overall orientation of the Dirichlet domain with respect to the sky The pattern of SI violation of a *cosmic topology leads to a measurable, characteristic Bipolar power spectrum* related to the principle directions in the Dirichlet domain and symmetries of the two point correlation function The Bipolar power spectroscopy of cosmic topology presents itself as promising pursuit for current and upcoming measurements of CMB anisotropy and polarization

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[36] *Global isotropy of space is violated in all multi-connected models (except, the 'Elliptical' space S^3/Z_2 [8])*

The same line of reasoning hold for CMB polarization when expressed in terms of Gaussian scalar $\tilde{E}(\hat{n})$ and pseudo-scalar $B(\hat{n})$ fields with corresponding two point correlation function For brevity, we explicitly discuss only the temperature anisotropy

[38] *This* … *the angular power spectrum,* $C_l = \frac{1}{2l+1}\sum_m |a_{lm}|^2$, *to reduce the cosmic variance*

[39] *For cubic torus the Dirichlet domain (DD) matches the fundamental domain (FD) However for torus spaces with cuboid and parallelpiped FD, the corresponding DD is very different, e g , hexahonal prism [18 25]*